

Handlebody subgroups in a mapping class group

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March 30, 2016

Abstract

Suppose subgroups $A, B < MCG(S)$ in the mapping class group of a closed orientable surface S are given and let $\langle A, B \rangle$ be the subgroup they generate. We discuss a question by Minsky asking when $\langle A, B \rangle \simeq A *_{A \cap B} B$ for handlebody subgroups A, B .

1 Introduction

Let V be a handlebody and $S = \partial V$ the boundary surface. We have an inclusion of mapping class groups, $MCG(V) < MCG(S)$. This subgroup is called a *handlebody subgroup* of $MCG(S)$. The kernel of the map $MCG(V) \rightarrow Out(\pi_1(V))$ is denoted by $MCG^0(V)$.

If $M = V_+ \cup_S V_-$ is a Heegaard splitting of a closed orientable 3-manifold, we have two handlebody subgroups $\Gamma_{\pm} = MCG(V_{\pm}) < MCG(S)$ with $S = \partial V_{\pm}$. Minsky [9, Question 5.1] asked

Question 1.1. *When is $\langle \Gamma_+, \Gamma_- \rangle < MCG(S)$ equal to the amalgamation $\Gamma_+ *_{\Gamma_+ \cap \Gamma_-} \Gamma_-$?*

Let $\mathcal{C}(S)$ be the curve graph of S and $D_{\pm} \subset \mathcal{C}(S)$ the set of isotopy classes of simple curves in S which bound disks in V_{\pm} . The *Hempel/Heegaard distance* of the splitting is defined to be equal to $d(D_+, D_-) = \min\{d_{\mathcal{C}(S)}(x, y) \mid$

*The first author was supported by the National Science Foundation grant 1308178. The second author is supported in part by Grant-in-Aid for Scientific Research (No. 23244005, 15H05739)

$x \in D_+, y \in D_-$. The group $MCG(S)$ acts on $\mathcal{C}(S)$ by isometries. The stabilizer subgroups of D_\pm in $MCG(S)$ are Γ_\pm . If the Hempel distance is sufficiently large, depending only on S (> 3 suffices if the genus of S is at least two [11]), then $\Gamma_+ \cap \Gamma_-$ is finite [16].

The following is the main result. It gives a (partial) negative answer to Question 1.1.

Theorem 5.1. *For the closed surface S of genus $4g + 1, g \geq 1$ and for any $N > 0$ there exists a Heegaard splitting $M = V_+ \cup_S V_-$ so that $\Gamma_+ \cap \Gamma_-$ is trivial, $\langle \Gamma_+, \Gamma_- \rangle$ is not equal to $\Gamma_+ * \Gamma_-$, and $d(D_+, D_-) \geq N$.*

We will prove Theorem 5.1 by constructing an example. To explain the idea we first construct a similar example in a certain group action on a simplicial tree (Theorem 4.1), then imitate it for the action of $MCG(S)$ on $\mathcal{C}(S)$.

By contrast, for the subgroups $MCG^0(V_\pm) = \Gamma_\pm^0 < \Gamma_\pm$, Ohshika-Sakuma [17] showed

Theorem 1.2. *If $d(D_+, D_-)$ is sufficiently large (depending on S), $\Gamma_+^0 \cap \Gamma_-^0$ is trivial and $\langle \Gamma_+^0, \Gamma_-^0 \rangle = \Gamma_+^0 * \Gamma_-^0$.*

That $\Gamma_+^0 \cap \Gamma_-^0$ is trivial follows from the fact that Γ_+^0, Γ_-^0 are torsion free (attributed to [19, proof of Prop 1.7] in [17]).

Here is an alternative proof, suggested by Minsky, that $\Gamma^0 < MCG(S)$ is torsion free for a handlebody V . Let $f \in \Gamma^0$ be a torsion element. Since f has finite order, we have a conformal structure on S invariant by f . Moreover, since f extends to V , by the classical deformation theory of Kleinian groups developed by Ahlfors, Bers, Kra, Marden, Maskit, and Sullivan, we have a unique hyperbolic structure on V whose conformal structure at infinity is the prescribed one. Since the conformal structure is f -invariant, so is the hyperbolic structure. Moreover, since f acts trivially on $\pi_1(V)$, each geodesic in V is invariant by f . This implies that f is identity on V , hence f is trivial.

Acknowledgements. We would like to thank Yair Minsky for useful comments.

2 Preliminaries

Let S be a closed orientable surface. The mapping class group $MCG(S)$ of S is the group of orientation preserving homeomorphisms modulo isotopy. The curve graph $\mathcal{C}(S)$ has a vertex for every isotopy class of essential simple closed curves in S , and an edge corresponding to pairs of simple closed curves that intersect minimally.

It is a fundamental theorem of Masur and Minsky [13] that the curve graph is δ -hyperbolic. Moreover, they show that an element F acts hyperbolically if and only if F is pseudo-Anosov, and that the translation length

$$trans(F) = \lim_{n \rightarrow \infty} \frac{d(x_0, F^n(x_0))}{n}, x_0 \in \mathcal{C}(S)$$

of F is uniformly bounded below by a positive constant that depends only on S . It follows that F has an invariant quasi-geodesic, called an *axis* denoted by $axis(F)$, whose quasi-geodesic constants depend only on S .

A subset $A \subset X$ in a geodesic space is *Q-quasi-convex* if any geodesic in X joining two points of A is contained in the Q -neighborhood of A . If S is the boundary of a handlebody V , then the set $D \subset \mathcal{C}(S)$ of curves that bound disks in V is quasi-convex (i.e. Q -quasi-convex for some Q), [14].

The stabilizer of the set D in $MCG(S)$ is $MCG(V)$, the mapping class group of the handlebody V , i.e., the group of isotopy classes of diffeomorphisms of V .

Given a Q -quasi-convex subset X in $\mathcal{C}(S)$, we define the nearest point projection $\mathcal{C}(S) \rightarrow X$. The nearest point projection is not exactly a map, but a *coarse map*, since for a given point maybe there is more than one nearest point, but the set of such points is bounded in diameter, and the bound depends only on δ and Q , but not on X .

In this paper we often take $axis(F)$ of a pseudo-Anosov element F as X . We may take any F -orbit instead of $axis(F)$. Two pseudo-Anosov elements F, G are independent (i.e., $\langle F, G \rangle$ is not virtually cyclic) if and only if the nearest point projection of $axis(E)$ to $axis(F)$ has a bounded image (cf. [13], [4]).

3 Acylindrical actions

In this section we discuss the acylindricity of a group action. This is a key property to prove Theorem 1.2.

Acylindricity was introduced by Sela for group actions on trees and extended by Bowditch [5]. Suppose G acts on a metric space X . The action is *acylindrical* if for given $R > 0$ there exist $L(R)$ and $N(R)$ such that for any points $v, w \in X$ with $|v - w| \geq L$, there are at most N elements $g \in G$ with $|v - g(v)|, |w - g(w)| \leq R$. (Here $|x - y|$ denotes the distance $d(x, y)$.) Bowditch [5] showed that the action of $MCG(S)$ on $\mathcal{C}(S)$ is acylindrical.

The following criterion will be useful.

Lemma 3.1. *Suppose G acts on a simplicial tree X . If the cardinality of the edge stabilizers is uniformly bounded then the action is acylindrical.*

Proof. Assume that every edge stabilizer contains at most K elements. Suppose an integer $R > 0$ is given. Take $L \gg R$. We will show that if $|v - w| \geq L$ then there are at most $(2R + 1)K$ elements g with $|v - gv|, |w - gw| \leq R$.

Indeed, let $[v, w]$ be the geodesic from v to w , and $[v, w]'$ and $[v, w]''$ be its subsegments after removing the R -neighborhood of v, w , and the $2R$ -neighborhood of v, w , respectively. Then by the assumption, $g([v, w]'') \subset [v, w]'$. Moreover, for an edge $E \subset [v, w]''$ near the midpoint, $g(E)$ is contained in $[v, w]'$ and the distance between E and $g(E)$ is at most R . Now fix such E . Then there are elements $h_1, \dots, h_n \in G'$ with $n \leq 2R + 1$, where $h_1 = 1$, such that for any concerned element g , there exists h_i with $h_i g(E) = E$. But since the stabilizer of E contains at most K elements, there are at most $nK \leq (2R + 1)K$ distinct choices for g . \square

To explain the background we quote a main technical result from [17] (we will not use this result).

Theorem 3.2. *Let a group G act acylindrically on a δ -hyperbolic space X . Then for a given $Q > 0$, there exists $M > 0$ with the following property. Let $A, B \subset X$ be Q -quasi-convex subsets, and $G_A < \text{stab}_G(A), G_B < \text{stab}_G(B)$ torsion-free subgroups. If $d_X(A, B) \geq M$ then*

- (1) $G_A \cap G_B$ is trivial.
- (2) $\langle G_A, G_B \rangle = G_A * G_B$.

Applying the theorem to the action of $MCG(S)$ on $\mathcal{C}(S)$ with $A = D_+, B = D_-$, and $G_A = \Gamma_+^0 < \Gamma_+ = \text{stab}(A), G_B = \Gamma_-^0 < \Gamma_- = \text{stab}(B)$, we obtain Theorem 1.2: if $d(D_+, D_-)$ is sufficiently large, depending only on S , then $\Gamma_+^0 \cap \Gamma_-^0$ is trivial and $\langle \Gamma_+^0, \Gamma_-^0 \rangle = \Gamma_+^0 * \Gamma_-^0$.

To explain the difference between the torsion-free setting of [17] and ours, we review the proof of Theorem 3.2. We start with an elementary lemma.

Lemma 3.3. *Let X be a δ -hyperbolic space and $A, B \subset X$ be Q -quasi-convex subsets. Let γ be a shortest geodesic between A and B . Then,*

- (1) *for any $x \in \gamma$ with both $d(x, A), d(x, B) > Q + 2\delta$, and for any shortest geodesic τ between A and B , we have $d(x, \tau) \leq 2\delta$.*
- (2) *Suppose f is an isometry of X with $f(A) = A, f(B) = B$. Then for any $x \in \gamma$ with both $d(x, A), d(x, B) > Q + 2\delta$, we have $d(x, f(x)) \leq 4\delta$. Hence for any $x \in \gamma$, we have $d(x, f(x)) \leq 2Q + 8\delta$.*
- (3) *Suppose f is an isometry of X with $f(A) = A$. For $x \in X \setminus A$ let σ be a shortest geodesic from x to A . For an integer $N > 0$ assume $d(x, A) \geq Q + 4\delta N$ and $d(x, f(x)) \leq 4\delta$. Then for any point $y \in \sigma$ with $Q + 2\delta < d(y, A) < d(x, A) - 4\delta N - 2\delta$, we have $d(y, f^i(y)) \leq 4\delta$ for $1 \leq i \leq N$.*

Proof. (1) Draw a geodesic quadrilateral with γ, τ a pair of opposite sides. By δ -hyperbolicity, x must be in the 2δ -neighborhood of one of the three sides not equal to γ , which must be τ , for otherwise, $d(x, A) \leq Q + 2\delta$ or $d(x, B) \leq Q + 2\delta$, impossible.

(2) Put $f(\gamma) = \tau$. Then for a point $x \in \gamma$ satisfying the assumption, by (1) there is a point $p \in f(\gamma)$ with $d(x, p) \leq 2\delta$. But $d(p, f(x)) \leq 2\delta$ since $d(x, A) = d(f(x), A)$ and $|d(x, A) - d(p, A)| \leq 2\delta$. By triangle inequality $d(x, f(x)) \leq 4\delta$. It then implies $d(x, f(x)) \leq 4\delta + 2(Q + 2\delta)$ for $x \in \gamma$ in general.

(3) By triangle inequality, for each $1 \leq i \leq N$, we have $d(x, f^i(x)) \leq 4\delta i$. Let $q = \sigma \cap A$, and draw a geodesic quadrilateral with the corners $x, q, f^i(q), f^i(x)$. Then by δ -hyperbolicity, a concerned point $y \in \sigma$ is in the 2δ -neighborhood of the side $f^i(\sigma)$, hence, as before $d(y, f^i(y)) \leq 4\delta$. \square

Proof of Theorem 3.2. (1) Set $L_0 = L(4\delta) + 2Q + 4\delta + 4\delta N(4\delta)$. We fix a constant $M \gg 2L_0$. Let γ be a shortest geodesic between A and B . Let $|\gamma|$ denote the length of γ . Since $|\gamma| \geq M \geq L_0$, we have points $x_1, x_2 \in \gamma$ such that $d(x_1, x_2) \geq L(4\delta)$ and all four of $d(x_1, A), d(x_1, B), d(x_2, A), d(x_2, B)$ are $> Q + 2\delta$. If $f \in G_A \cap G_B$, then both $d(x_1, f(x_1)), d(x_2, f(x_2)) \leq 4\delta$ by Lemma 3.3 (2), hence by the acylindricity there are at most $N(4\delta)$ such elements, so the order of $G_A \cap G_B$ is $\leq N(4\delta)$. In particular each element in $G_A \cap G_B$ is torsion. Since G is torsion free, $G_A \cap G_B$ is trivial.

(2)

Claim 1. Let $1 \neq f \in G_A$. If $d(x, A) \geq L_0$ then $d(x, f(x)) > 4\delta$.

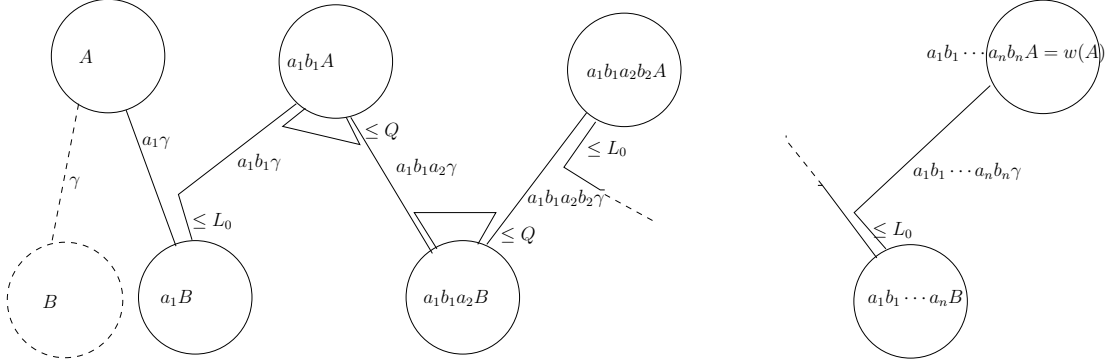


Figure 1: There is a piecewise geodesic from A to $w(A)$, connecting $a_1\gamma, a_1b_1\gamma, \dots, a_1b_1 \dots a_nb_n\gamma$ in this order, whose length is at least $|w||\gamma|$. Since the backtrack at each connecting point is $\leq L_0$, the path is a quasi-geodesic, say $(1.1, 2L_0)$ -quasi-geodesic, since $|\gamma| \gg L_0$. In fact its length roughly gives a lower bound of the distance between A and $w(A)$.

To argue by contradiction assume $d(x, f(x)) \leq 4\delta$. Let σ be a shortest geodesic from x to A . Apply Lemma 3.3 (3) with $N = N(4\delta)$. Then for each point $y \in \sigma$ with $Q + 2\delta < d(y, A) < d(x, A) - 4\delta N(4\delta) - 2\delta$ and each $1 \leq i \leq N$, we have $d(y, f^i(y)) \leq 4\delta$. Now the subsegment of σ , except for the end points, that those y can belong to has length at least $L_0 - 4\delta N(4\delta) - 2\delta - (Q + 2\delta) = L(4\delta) + Q$. Taking two points near each end of the subsegment, they are moved at most 4δ by $1, f, \dots, f^N$. But by acylindricity there are at most $N(4\delta)$ such elements. Since $N = N(4\delta)$, the order of f must be at most $N(4\delta)$, hence trivial, contradiction.

It follows from Claim 1 that if $x \in \gamma$ with $d(x, A) \geq L_0$, then x is not contained in $N_{2\delta}(f(\gamma))$.

Similarly,

Claim 2. Let $1 \neq f \in G_B$. If $d(x, B) \geq L_0$ then $d(x, f(x)) > 4\delta$.

Notice that Claim 1 and Claim 2 hold if every non-trivial element in G_A, G_B has order at least $N(4\delta) + 1$ or ∞ .

Now, we apply $1 \neq a \in G_A$ to $A \cup \gamma \cup B$, and obtain $a(B) \cup a(\gamma) \cup A \cup \gamma \cup B$. Put $p = \gamma \cap A$. The path $a(\gamma) \cup [a(p), p] \cup \gamma$ is roughly a shortest geodesic from aB to B . This is because $|\gamma| \geq M \gg 2L_0$, Claim 1, and that the geodesic $[a(p), p]$ is contained in the Q -neighborhood of A . So, $d(aB, B)$ is at least, say, $2(|\gamma| - L_0 - 10\delta)$. Similarly, now using Claim 2, for any $1 \neq b \in G_B$, $d(bA, A)$ is at least $2(|\gamma| - L_0 - 10\delta)$.

To finish, given a reduced word in $G_A * G_B$, $w = a_1 b_1 \cdots a_n b_n$, we let the elements $b_n, a_n, \dots, b_1, a_1$ successively act on A (or B if b_n is empty). See Figure 1. Then as before the distance between A and $w(A)$ is at least, say, $|w|(|\gamma| - 2L_0 - 10\delta)$, where $|w|$ is the length as a reduced word. (Here we are using a standard fact in δ -hyperbolic geometry that a piecewise geodesic with each geodesic part long and the “backtrack” at each connecting point short is not only a quasi-geodesic, but also a geodesic with the same endpoints follows the path except for the backtrack parts.) In particular $A \neq w(A)$, so w is not trivial in G . It implies $\langle G_A, G_B \rangle = G_A * G_B$. \square

There is a more general version of Theorem 3.2 in which one does not assume that G_A and G_B are torsion free. To state it we introduce the following definition. For an isometry $f : X \rightarrow X$ define the *coarse fixed set* as $CFix(f) = \{x \in X \mid d(x, f(x)) \leq 4\delta\}$.

Also, we will need a version for more than two subsets in X to discuss another application. For that we introduce one more definition. Let A_1, A_2, \dots, A_n be mutually disjoint subsets in a δ -hyperbolic space X . For a given constant $K > 0$, we say that A_i is *K-terminal* if for any other A_j, A_k and any shortest geodesic γ between A_j, A_k , the distance between A_i and γ is at least K .

Theorem 3.4. *Let a group G act acylindrically on a δ -hyperbolic space X . Then for a given $Q > 0$, there exists $K > 0$ with the following property. Let $A_1, \dots, A_n \subset X$ be Q -quasi-convex subsets, and $G_{A_i} < \text{stab}_G(A_i)$ be subgroups for all i .*

Assume that there exists a subset A'_i containing A_i for each i so that:

- (a) *for every finite order element $1 \neq a \in G_{A_i}$ we have $CFix(a) \subset A'_i$ for each i ;*
- (b) *$d_X(A'_i, A'_j) \geq K$ for all pairs $i \neq j$; and*
- (c) *each A'_i is K -terminal.*

Then

- (1) *$G_{A_i} \cap G_{A_j}$ is trivial for all $i \neq j$.*
- (2) *$\langle G_{A_1}, \dots, G_{A_n} \rangle = G_{A_1} * \dots * G_{A_n}$.*

The proof is a slight variation of the proof of Theorem 3.2 and is omitted. If G_{A_i} does not contain any non-trivial elements of finite order, then we just

put $A_i = A'_i$. Our counterexamples will have the property that $A'_1 \cap A'_2 \neq \emptyset$ (namely, the two coarse fixed sets intersect although A_1 and A_2 are far away. cf. Claim 1 in the proof of Theorem 3.2 (2) when f has infinite order, where $C\text{Fix}(f)$ is contained in the L_0 -neighborhood of A).

If the sets A'_1, \dots, A'_n satisfy properties (b) and (c) above for a constant K , we say that they are *K-separated*.

4 Example on a tree

We will show that Theorem 3.2 does not hold if we do not assume that G_A and G_B are torsion-free. We construct a counterexample in the action of $MCG(S)$ on $\mathcal{C}(S)$ (Theorem 5.1).

To explain the idea we start with a counterexample when X is a simplicial tree. The key geometric feature is that, if we keep the previous notations, γ and $a(\gamma)$ may stay close along an arbitrarily long segment if a has finite order (each point on that segment does not move very much by a).

Theorem 4.1. *There exists an acylindrical group action on a simplicial tree X by a group G such that for any number $N > 0$ there exist vertices $v, w \in X$ with $|v - w| \geq N$ such that $\text{stab}_G(v) \cap \text{stab}_G(w)$ is trivial and $\langle \text{stab}_G(v), \text{stab}_G(w) \rangle$ is not equal to the free product $\text{stab}_G(v) * \text{stab}_G(w)$.*

Proof. We first construct an example with $N = 2$. Start with abelian groups A, B with non-trivial torsion elements $a \in A$ and $b \in B$, for example, $A, B \simeq \mathbb{Z}/2\mathbb{Z}$.

Define the group

$$G = A *_{\langle a \rangle} (\langle a \rangle \times \langle b \rangle) *_{\langle b \rangle} B$$

and let T be the Bass-Serre tree of this graph of groups decomposition.

There are two vertices v, w in T at distance two whose stabilizers are A and B . The intersection $A \cap B$ is trivial in G since $\langle a \rangle \cap \langle b \rangle$ is trivial in $\langle a \rangle \times \langle b \rangle$. On the other hand, $\langle A, B \rangle = G$ is not equal to $A * B$ since G is the quotient of the free product $A * B$ by the relation $ab = ba$. The geometric reason for why $\langle a, b \rangle$ is not equal to $\langle a \rangle * \langle b \rangle$ is that $\text{Fix}(a)$ and $\text{Fix}(b)$ intersect non-trivially in T .

The action on T is acylindrical by Lemma 3.1 since the edge stabilizer is a conjugate of $\langle a \rangle$ or $\langle b \rangle$.

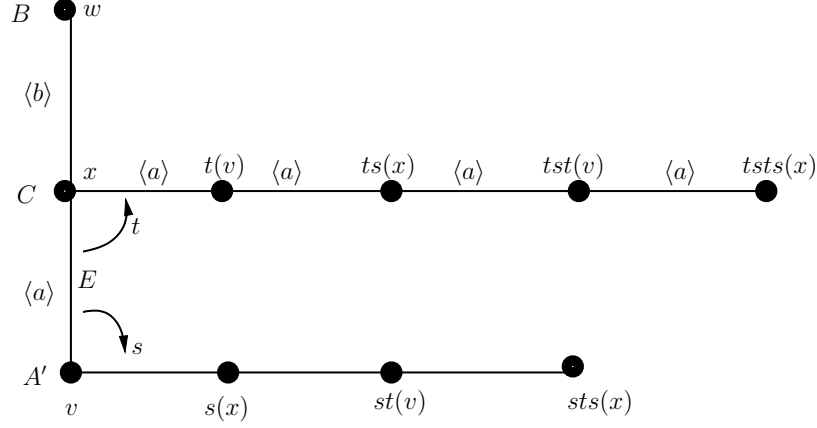


Figure 2:

To produce an acylindrical action that works for all $N > 1$ we modify the previous example. Take the direct product of $\mathbb{Z} = \langle t \rangle$ and the subgroup $\langle a \rangle \times \langle b \rangle$ in G . One can write the new group as

$$A *_{\langle a \rangle} \{ \langle a \rangle \times \langle b \rangle \times \langle t \rangle \} *_{\langle b \rangle} B$$

Further, add a new element s to A with a relation $sa = as$ to get $A' = A \times \langle s \rangle$ and set $C = \langle a \rangle \times \langle b \rangle \times \langle t \rangle$ and

$$G' = A' *_{\langle a \rangle} C *_{\langle b \rangle} B$$

This is a two edge decomposition.

In the Bass-Serre tree of this decomposition, consider the “fundamental domain”, i.e. the subtree spanned by two vertices v, w at distance two with stabilizers A' and B respectively. Let x be the vertex between them with stabilizer C . See Figure 2.

Now consider the ray based at x that contains the vertices $x, t(v), ts(x), tst(v), tsts(x), \dots$. The stabilizer of every edge on this ray is $\langle a \rangle$ since both t and s commute with a .

So, the intersection of B , the vertex group of w , and any of the vertex groups along the ray except for C is $\langle a \rangle \cap \langle b \rangle = 1$.

But for each $n > 0$ the subgroup $\langle C^{(ts)^n}, B \rangle < G'$ is not equal to $C^{(ts)^n} * B$ since $a \in C^{(ts)^n}$ and $b \in B$ generate $\langle a \rangle \times \langle b \rangle$ and not $\langle a \rangle * \langle b \rangle$.

The action of G' is acylindrical by Lemma 3.1 since any edge stabilizer is a conjugate of $\langle a \rangle$ or $\langle b \rangle$. \square

5 Example on $\mathcal{C}(S)$ and proof of theorem

We will prove the main theorem.

Theorem 5.1. *For the closed surface S of genus $4g + 1$ with $g \geq 1$ and for any $N > 0$ there exists a Heegaard splitting $M = V_+ \cup_S V_-$ so that $\Gamma_+ \cap \Gamma_-$ is trivial, $\langle \Gamma_+, \Gamma_- \rangle$ is not equal to $\Gamma_+ * \Gamma_-$, and $d(D_+, D_-) \geq N$.*

We will need two properties of pseudo-Anosov elements to prove the theorem (Lemma 5.2, Lemma 5.4).

5.1 Pseudo-Anosov elements by Masur-Smillie

Let S be a closed surface and F a pseudo-Anosov mapping class on S . The *elementary closure* of F is the subgroup $E(F)$ of $MCG(S)$ that consists of mapping classes preserving the stable and unstable foliations of F . Equivalently, $E(F)$ is the centralizer of F in $MCG(S)$. The group $E(F)$ contains a unique finite normal subgroup $N(F)$ such that $E(F)/N(F)$ is infinite cyclic. Note that $E(F^k) = E(F)$ and $N(F^k) = N(F)$ for every $k \neq 0$.

If $S' \rightarrow S$ is a regular cover with deck group Δ and if $F : S \rightarrow S$ is a pseudo-Anosov mapping class with $N(F) = 1$, then we certainly have $N(F') \supseteq \Delta$ for any lift $F' : S' \rightarrow S'$ of any power of F , but strict inclusion may hold. It is an interesting question whether one can construct F so that equality holds for all regular covers. We call such F *prime* and we discuss a construction of prime pseudo-Anosov mapping classes in Section 6. For our purposes we need quite a bit less.

Lemma 5.2. *Suppose $F : S \rightarrow S$ is a pseudo-Anosov mapping class whose stable and unstable foliations have two singular points, one of order p , the other of order q , with both p, q odd and relatively prime. Let $S' \rightarrow S$ be a double cover with deck group $\langle a \rangle$ and F' a lift of a power of F to S' . Then $N(F) = 1$ and $N(F') = \langle a \rangle$.*

Proof. We first argue that $N(F) = 1$. Suppose $g \in N(F)$. Then g can be represented by a homeomorphism, also denoted $g : S \rightarrow S$, that preserves both measured foliations. In particular, g is an isometry in the associated flat metric on S with cone type singularities. The homeomorphism g fixes both singular points and satisfies both $g^p = 1$ and $g^q = 1$, since an isometry that fixes a nonempty open set is necessarily the identity. Since p, q are relatively prime it follows that $g = 1$.

We now argue that $N(F') = \langle a \rangle$. We have $\langle a \rangle < N(F')$. Let $g \in N(F')$, then $g : S' \rightarrow S'$ is a finite order homeomorphism that preserves the lift of stable and unstable foliations of F . Composing with a if necessary we may assume that g fixes both p -prong singularities. Arguing as above, we see that $g^p = 1$. Since g^2 fixes both q -prong singularities, similarly we have $g^{2q} = 1$ and since $(p, 2q) = 1$ we have $g = 1$. We showed $N(F') = \langle a \rangle$. \square

Corollary 5.3. *Pseudo-Anosov mapping classes as in Lemma 5.2 exist in every genus ≥ 3 .*

Proof. Write $4g = p + q$ where p, q are relatively prime odd numbers. For example, we can take $p = 2g - 1$ and $q = 2g + 1$. By the work of Masur-Smillie [15] F as above exists. \square

5.2 Masur domain and Hempel elements

Suppose V is a handlebody and S its boundary. Let $D \subset \mathcal{C}(S)$ be the set of curves that bound disks in V . Denote by $L \subset \mathcal{PML}(S)$ the closure of D , viewed as a subset of $\mathcal{PML}(S)$. Then L is nowhere dense in $\mathcal{PML}(S)$ [12], and its complement Ω is called the *Masur domain*.

Hempel [10] found that if the stable lamination of a pseudo-Anosov element F is in Ω then $\lim_{n \rightarrow \infty} d_{\mathcal{C}(S)}(D, F^n(D)) = \infty$. We say a pseudo-Anosov element $F : S \rightarrow S$ is *Hempel* for D if the nearest point projection of D to $axis(F)$ is a bounded set.

F is Hempel if and only if the end points of $axis(F)$ are in Ω , [3]. On the other hand, both endpoints of $axis(F)$ are in L if and only if $axis(F)$ is contained in a K -neighborhood of D for some $K > 0$ since both D and $axis(F)$ are quasi-convex subsets in the δ -hyperbolic space $\mathcal{C}(S)$ (cf. [3]). Since L is nowhere dense in $\mathcal{PML}(S)$ and the set of pairs of endpoints (λ^+, λ^-) of pseudo-Anosov mapping classes is dense in $\mathcal{PML}(S) \times \mathcal{PML}(S)$, there is a pseudo-Anosov element F whose stable and unstable laminations are not in L , so that F is Hempel.

Masur found a condition in terms of the intersection number for a curve to be in D [12, Lemma 1.1] and used it to prove L is nowhere dense [12, Theorem 1.2]. The following lemma is proved using his ideas.

Lemma 5.4. *Let $V' \rightarrow V$ be a double cover between handlebodies with the deck group $\langle a \rangle$, $S' = \partial V'$, $S = \partial V$, and $D' \subset \mathcal{C}(S')$, $D \subset \mathcal{C}(S)$ the set of curves that bound disks in V', V , respectively.*

If the genus of S is ≥ 3 , then $MCG(S)$ contains a pseudo-Anosov element F such that:

- (i) F is Hempel for D ,
- (ii) F lifts to $F' : S' \rightarrow S'$ and F' is Hempel for D' ,
- (iii) $N(F') = \langle a \rangle$.

Proof. Let Ω be the Masur domain for V and Ω' for V' . We first find a lamination Λ on S that is in Ω such that its lift Λ' on S' is also in Ω' . Choose a pants decomposition of S using curves in D and a lamination $\Lambda \in \mathcal{PML}(S)$ whose support intersects each pair of pants in this decomposition in 3 (non-empty) families of arcs connecting distinct boundary components (so there are no arcs connecting a boundary component to itself). In the proof of [12, Theorem 1.2] Masur shows that $\Lambda \in \Omega$ (for example, take the curve β in his proof as Λ). This is done by verifying the conditions in Lemma 1.1 for β with respect to the pants decomposition in the last two paragraphs of the proof of Theorem 1.2. Now the lift Λ' of Λ to S' satisfies the same condition with respect to the lifted pants decomposition (it lifts since our covering is between handlebodies and the boundary curves bound disks), so we have $\Lambda' \in \Omega'$.

Choose a pseudo-Anosov homeomorphism $G : S \rightarrow S$ both of whose fixed points in $\mathcal{PML}(S)$ are close to Λ and in particular they are in Ω since Ω is open. The lift G' of G (or its power) to S' similarly has endpoints close to Λ' and in particular in Ω' . It follows that both G and G' are Hempel.

To finish the proof we need to arrange that G has the extra property (iii). Let $H : S \rightarrow S$ be an arbitrary pseudo-Anosov mapping class that satisfies the assumption of Lemma 5.2. Such H exists by Corollary 5.3. Then $F = G^n H G^{-n}$ also satisfies the assumptions, and hence also conclusion of Lemma 5.2 for any $n > 0$ and has an axis whose endpoints are close to Λ if $n > 0$ is sufficiently large. Therefore F is Hempel, and similarly, the lift F' has an axis whose endpoints close to Λ' , therefore F' is Hempel. \square

5.3 Proof of Theorem 5.1

We prove Theorem 5.1 by constructing an example.

Proof of Theorem 5.1. Let $H \simeq \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$ with generators a_1, a_2 , and let $V' \rightarrow V$ be a normal cover between handlebodies with the deck group H . If

$g \geq 2$ is the genus of V , then the genus of V' is $4g - 3$. Let $D' \subset \mathcal{C}(S')$ be the set of curves in $S' = \partial V'$ that bound disks in V' . We have two double covers $S' \rightarrow S'/a_i$. Let $D_i \subset \mathcal{C}(S'/a_i)$ be the set of curves in S'/a_i that bound disks in V'/a_i . Put $S_i = S'/a_i$. The genus of S_i is $2g - 1$. Let Q be a common quasi-convex constant for D', D_1, D_2 , and δ the hyperbolicity constant of $\mathcal{C}(S')$.

Using Lemma 5.4, take a pseudo-Anosov element F_i on S_i that is Hempel for D_i such that the lift F'_i of F_i to S' is also Hempel for D' , and that $N(F'_i) = \langle a_i \rangle$. Note that F'_1, F'_2 are independent pseudo-Anosov elements on S' since their elementary closures are different. In particular, the projection of $axis(F'_1)$ to $axis(F'_2)$ is bounded, and vice versa.

Note that $a_i \in stab(D')$ since $a_i \in H$. Set $D'_i = F_i'^N(D')$ for $N > 0$. Then, $a_i \in stab(D'_i)$ since F'_i centralizes a_i .

Form the Heegaard splitting $V'_+ \cup_{S'} V'_-$ such that $D_+ = D'_1, D_- = D'_2 \subset \mathcal{C}(S')$. The surface S' is fixed but the splitting depends on N . We will argue this is a desired splitting.

Set $\Gamma_i = stab(D'_i) < MCG(S')$. In other words, $\Gamma_1 = \Gamma_+, \Gamma_2 = \Gamma_-$ in the Heegaard splitting convention. Since $a_i \in \Gamma_i$ and $a_1 a_2 = a_2 a_1$, $\langle \Gamma_1, \Gamma_2 \rangle$ is not the free product of Γ_1, Γ_2 .

To prove the theorem we are left to verify $d(D'_1, D'_2) \rightarrow \infty$ as $N \rightarrow \infty$ and $\Gamma_1 \cap \Gamma_2 = 1$ for any large $N > 0$.

Lemma 5.5. $d(D'_1, D'_2) \rightarrow \infty$ as $N \rightarrow \infty$.

Proof. We claim that there is a constant A such that for any $N > 0$,

$$d(D'_1, D'_2) \geq (trans(F'_1) + trans(F'_2))N - A.$$

Let π_1 denote the projection to $axis(F'_1)$, and π_2 the projection to $axis(F'_2)$. As we said they are coarse maps but we pretend they are maps for simplicity. Also, we pretend that both $axis(F'_1), axis(F'_2)$ are geodesics.

Let L be a common bound of the diameter of the sets $\pi_1(D')$, $\pi_1(axis(F'_2))$, $\pi_2(D')$ and $\pi_2(axis(F'_1))$. Then L is a bound of $\pi_1(D'_1)$ and $\pi_2(D'_2)$ for all $N > 0$. Choose points $q_1 \in \pi_1(D')$, $q_2 \in \pi_2(D')$, $r_1 \in \pi_1(axis(F'_2))$ and $r_2 \in \pi_2(axis(F'_1))$.

Now assume $N > 0$ is so large that $\pi_1(axis(F'_2))$ and $\pi_1(D'_1)$ are far apart, and also $\pi_2(axis(F'_1))$ and $\pi_2(D'_2)$ are far apart (compared to L and δ). It suffices to show the above inequality under this assumption.

Let $y_1 \in D'_1$ and $y_2 \in D'_2$ be any points, and put $x_1 = \pi_1(y_1), x_2 = \pi_2(y_2)$. Then, by a standard argument using δ -hyperbolicity, the piecewise

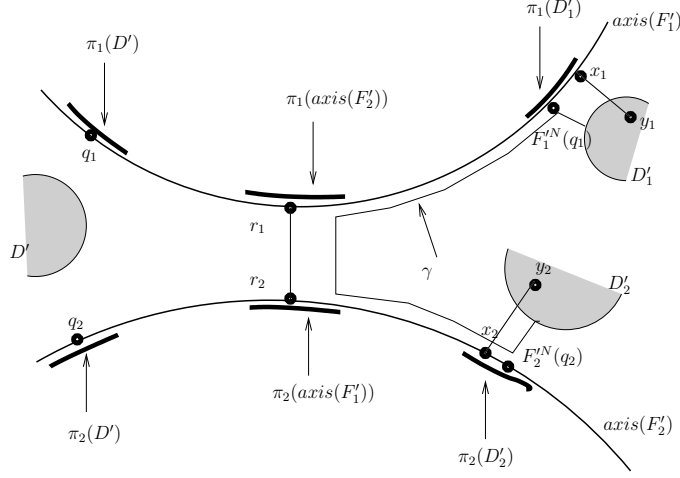


Figure 3: If $N > 0$ is large, the projection $\pi_1(\text{axis}(F'_2))$ and $\pi_1(D'_1)$ are far apart on $\text{axis}(F'_1)$. Also, $\pi_2(\text{axis}(F'_1))$ and $\pi_2(D'_2)$ are far apart on $\text{axis}(F'_2)$. As a consequence any shortest geodesic γ between D'_1 and D'_2 must enter a bounded neighborhood of each of those four projection sets, and the segment in γ near $\text{axis}(F'_i)$ is almost fixed by a_i pointwise.

geodesic $[y_1, x_1] \cup [x_1, r_1] \cup [r_1, r_2] \cup [r_2, x_2] \cup [x_2, y_2]$ is a quasi-geodesic with uniform quasi-geodesic constants that depends only on L and δ . See Figure 3. Hence the Hausdorff distance between the piecewise geodesic and the geodesic $[y_1, y_2]$ is bounded (the bound depends only on L and δ).

It follows that there is a constant C such that $d(D'_1, D'_2) \geq d(y_1, x_1) + d(x_1, r_1) + d(r_1, r_2) + d(r_2, x_2) + d(x_2, y_2) - C \geq d(x_1, r_1) + d(r_2, x_2) - C \geq d(F_1'^N(q_1), r_1) - L + d(F_2'^N(q_2), r_2) - L - C$. On the other hand since $q_1, r_1 \in \text{axis}(F'_1)$ and $q_2, r_2 \in \text{axis}(F'_2)$ there is a constant B such that for all $N > 0$ we have $d(F_1'^N(q_1), r_1) + d(F_2'^N(q_2), r_2) \geq (\text{trans}(F'_1) + \text{trans}(F'_2))N - B$. Combining them we get a desired estimate with $A = 2L + B + C$. \square

We note that any geodesic joining a point in D'_1 and a point in D'_2 passes through a bounded neighborhood of each of $F_1'^N(q_1), r_1, r_2, F_2'^N(q_2)$ provided that $N > 0$ is large enough. The bound depends only on L and δ . See Figure 3. In the argument we did not use that D' (as well as D'_1, D'_2) are quasi-convex.

To argue $\Gamma_1 \cap \Gamma_2 = 1$, we will need the following lemma from [4, Proposition 6]. This is a consequence of the fact that F is a “WPD element”.

Lemma 5.6. *Let F be a pseudo-Anosov mapping class on a hyperbolic surface S . There is a constant $M > 0$ such that for any $g \in MCG(S)$ the diameter of the projection of $g(\text{axis}(F))$ to $\text{axis}(F)$ in $\mathcal{C}(S)$ is larger than L , then $g \in E(F)$.*

Lemma 5.7. $\Gamma_1 \cap \Gamma_2 = 1$ for any large $N > 0$.

Proof. By Lemma 5.5 choose N large such that $d(D'_1, D'_2)$ is very large compared to δ and L . Let γ be a shortest geodesic from D'_1 to D'_2 . Then as we noted after the proof of Lemma 5.5, γ passes through the bounded neighborhood of each of $F_1'^N(q_1), r_1, r_2, F_2'^N(q_2)$.

Now let $f \in \Gamma_1 \cap \Gamma_2$. Then we have $d(x, f(x)) \leq 2Q + 8\delta$ for any $x \in \gamma$ by Lemma 3.3. Since all of $F_1'^N(q_1), r_1, r_2, F_2'^N(q_2)$ are in bounded distance from γ we conclude each of those four points is moved by f a bounded amount (the bound depends only on δ, L, Q).

But since $F_1'^N(q_1), r_1$ are contained in $\text{axis}(F'_1)$ and are far apart from each other for any large $N > 0$, we find $f(\text{axis}(F'_1))$ has a long ($> M$) projection to $\text{axis}(F'_1)$, hence $f \in E(F'_1)$ by Lemma 5.6. By the same reason $f \in E(F'_2)$. We conclude $f \in E(F'_1) \cap E(F'_2)$. But since F'_1 and F'_2 are independent, f must be a torsion element, so $f \in N(F'_1) \cap N(F'_2)$. By Lemma 5.2, $f \in \langle a_i \rangle \cap \langle a_2 \rangle = 1$. We showed the lemma. \square

We proved the theorem. \square

6 Prime pseudo-Anosov elements

In view of Lemma 5.2 we introduce a property that looks interesting for its own sake. We say a pseudo-Anosov mapping class F is *prime* if its stable/unstable foliations are not lifts of any foliations of a (possibly orbifold) quotient of S .

If F is prime then $E(F)$ is cyclic and $N(F) = 1$. Indeed, if $N(F) \neq 1$ then the two foliations lift from $S/N(F)$, with $N(F)$ realized as a group of isometries of S using Nielsen realization. Moreover,

Lemma 6.1. *(cf. Lemma 5.2) Suppose $S' \rightarrow S$ is a finite cover with the Deck group Δ . Let F be a prime pseudo-Anosov element on S and F' a lift of a power of F to S' . Then $N(F') = \Delta$.*

Proof. It is clear that $\Delta < N(F')$. If the inclusion is strict, then the stable and unstable foliations of F can be obtained by pulling back from $S'/N(F') = S/(N(F')/\Delta)$. So we have a contradiction. \square

Note that if we have a prime pseudo-Anosov element on S , we can use Lemma 6.1 instead of Lemma 5.2 in the proof of Lemma 5.4 and Theorem 5.1. We will give a construction of prime pseudo-Anosov elements when the genus of S is 3, so this will also prove the theorem for the genus 5 case.

Recall that if a, b, c, d are 4 vectors in \mathbb{R}^2 then the cross ratio is

$$[a, b; c, d] = \frac{[a, c][b, d]}{[a, d][b, c]}$$

where $[x, y] = x_1y_2 - x_2y_1$ for $x = (x_1, x_2), y = (y_1, y_2)$. We do not define it when one of $[a, c], [b, d], [a, d], [b, c]$ is 0. The cross ratio is invariant under changing signs and scaling individual vectors and applying matrices in $SL_2(\mathbb{R})$. It follows that for any flat structure on the torus the cross ratio for the vectors in the directions of four distinct closed geodesics is (well-defined and) rational.

A singular Euclidean structure (or just a *flat* structure) on a surface S is *good* if the cone angle is a multiple of π at each singularity. A geodesic segment connecting two singular points, or a closed geodesic is *good* if the angle along the geodesic at each singular point is a multiple of π .

The developing map $\widetilde{S - \Sigma} \rightarrow \mathbb{R}^2$ defined on the universal cover of the complement of the cone points will take a good geodesic to a straight line, or a line segment. So, for any four good geodesics, the cross ratio for the four directions, if they are distinct, is well-defined.

Next, if $S' \rightarrow S$ is a branched cover between good flat structures, then the cross ratio of four good geodesics in S' is equal to the cross ratio of their images in S , simply because S, S' have the “same” developing map. In particular, all cross ratios between good geodesics on a torus or a sphere with 4 cone points are rational, and to prove that a particular good flat surface is not commensurable with a torus it suffices to produce four good geodesics whose cross ratio is irrational.

Lemma 6.2. *Suppose $F : S \rightarrow S$ is a pseudo-Anosov homeomorphism such that:*

- (1) *the stable foliation of F has two singular points x and y , with p and q prongs respectively, and with p and q distinct odd primes, and*

(2) *a flat structure on S determined by the stable and unstable foliations has four good closed geodesics with the cross ratio of their (distinct) direction vectors $a, b, c, d \in \mathbb{R}^2$ irrational.*

Then F is prime.

We note that there is a 2-parameter family of flat structures determined by the two foliations; they depend on the choice of the transverse measure in a projective class on each foliation. However, since scaling and linear transformations do not change the cross ratio, the assumption is independent of these choices.

Proof. Let p, q, F be as in the statement. Now suppose $\pi : S \rightarrow S'$ is a branched cover of degree $d > 1$ and $\mathcal{F} = \pi^{-1}\mathcal{F}'$. The local degree of π at x is either 1 or p . It cannot be 1, since at any other preimage of $\pi(x)$ the singularity would have to have kp prongs, and there aren't any. Thus at x the map is modeled on $z \mapsto z^p$, and similarly at y it looks like $z \mapsto z^q$. There are now two cases.

Case 1. $\pi(x) \neq \pi(y)$.

It follows that the other points that map to $\pi(x)$ have 2 prongs and so the map there has local degree 2. Thus d is odd and away from the images of singular points the foliation \mathcal{F}' is regular (since otherwise d would have to be even). Thus there are $(d-p)/2$ other preimages of $\pi(x)$, and deleting these and the same for the q -prong singularity we get that the Euler characteristic of $S - \pi^{-1}(\{\pi(x), \pi(y)\})$ is

$$(2 - 2g) - (d-p)/2 - (d-q)/2 - 2 = -d$$

So the Euler characteristic of the quotient S' minus 2 singular points is -1 , i.e. the quotient is the twice punctured $\mathbb{R}P^2$, which does not support any pseudo-Anosov homeomorphisms (e.g. the curve complex is finite, see [20]). On the other hand, let \tilde{S} be a finite cover of the punctured S so that the induced cover to $\mathbb{R}P^2$ minus two points is regular. Some power of F lifts to \tilde{F} on \tilde{S} , and since the cover is regular, a further power of \tilde{F} descends to $\mathbb{R}P^2$ (since each element, a , of the Deck group leaves the stable and unstable foliations invariant, so that $(a\tilde{F}a^{-1})^N = \tilde{F}^N$ for some $N > 0$, so a and \tilde{F}^N commute), contradiction.

Case 2. $\pi(x) = \pi(y)$.

Again the other points that map to $\pi(x) = \pi(y)$ are regular and the map has local degree 2, so there are $\frac{d-(p+q)}{2}$ such points. Here d is even and we

may have some number, say $k \geq 0$, of 1-prong singularities z_1, \dots, z_k in the quotient, with each singularity having $\frac{d}{2}$ preimages where local degree is 2. Now we have that the Euler characteristic of $S - \pi^{-1}(\{\pi(x), z_1, \dots, z_k\})$ is

$$(2 - 2g) - \frac{d - (p + q)}{2} - 2 - \frac{kd}{2} = -\frac{(k + 1)d}{2}$$

So k is odd and the Euler characteristic of the quotient S' minus the singular points is $-\frac{k+1}{2}$. So, the Euler characteristic of S' is $-\frac{k+1}{2} + (k+1) = \frac{k+1}{2}$. The only possibilities are $k = 1$ and $k = 3$, and the quotient is twice punctured $\mathbb{R}P^2$ or 4 times punctured S^2 . The first possibility is ruled out as in Case 1.

In the second case the good flat structure on S descends to a good flat structure on S^2 with 4 singular points, then lifts to a flat structure on the branch double cover T^2 , with four closed geodesics such that the cross ratio of the four direction vectors is $[a, b; c, d]$ that is not rational, contradiction.

Indeed, using the same notation as in Case 1, a power of F lifts to \tilde{F} on \tilde{S} that regularly covers S^2 minus 4 points. We lift the flat structure on S and the stable and unstable foliations of F to \tilde{S} . Then their regular leaves are straight lines. Each deck transformation preserves the foliations, so that it is an isometry of \tilde{S} , and that the good flat structure on \tilde{S} , with cone angle at each singular point at least 2π , descends to a good flat structure of S^2 minus 4 points (and the angle at each puncture is π). We obtain a good flat structure on S^2 with four good closed geodesics and the cross ratio is $[a, b; c, d]$. Also, the cross ratio will not change when we take a double cover that is a flat torus, contradiction. \square

Remark 6.3. *Regarding the assumption (1), if g is the genus of S , by an Euler characteristic count we must have $p+q = 4g$. Conversely, the Goldbach conjecture predicts that every even integer > 2 can be written as a sum of two primes. When the integer is ≥ 8 and divisible by 4, the two primes are necessarily distinct and odd. For example, $12 = 5 + 7$ satisfies the Goldbach conjecture. The work of Masur-Smillie [15] shows that if $g \geq 3$ and $4g = p+q$ then the surface S admits a pseudo-Anosov homeomorphism whose stable and unstable foliations have two singular points, one of order p , the other of order q .*

Example 6.4. *We now construct an explicit example in genus 3 satisfying the assumption of Lemma 6.2. We take $p = 5$, $q = 7$. Consider the flat square tiled surface S pictured below. Edges labeled by the same letter are*

to be identified. If the edges are on opposite sides of the parallelogram they are identified by a translation, and otherwise by a rotation by π . The square tiling of \mathbb{R}^2 induces one on the surface S . There are two cone points, with cone angles 5π and 7π respectively. So, S has a good flat structure.

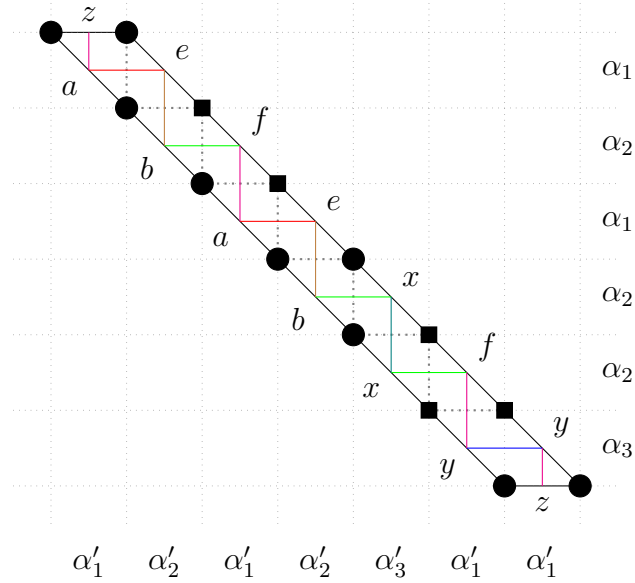


Figure 4: The square tiled surface. The round vertex has 7 prongs and the square vertex has 5. The surface has a good flat structure.

We will use Thurston's construction of pseudo-Anosov homeomorphisms [7, Theorem 14.1] to construct F . The lines bisecting the squares form three horizontal and three vertical geodesics. The matrix N of intersection numbers, where the jk entry is the intersection number $i(\alpha_j, \alpha'_k)$, is

$$N = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The largest eigenvalue of NN^t is $\mu = 5.0489\dots$ satisfying the minimal polynomial $\mu^3 - 6\mu^2 + 5\mu - 1 = 0$. Then one can choose the lengths and heights of the squares (making them into rectangles, which gives a tiling of

S) so that the twist in the horizontal multicurve is given by the matrix

$$\begin{pmatrix} 1 & \mu^{1/2} \\ 0 & 1 \end{pmatrix}$$

and the twist in the vertical multicurve by the matrix

$$\begin{pmatrix} 1 & 0 \\ -\mu^{1/2} & 1 \end{pmatrix}$$

This means that the product of the first and the inverse of the second is

$$\begin{pmatrix} 1 & \mu^{1/2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mu^{1/2} & 1 \end{pmatrix} = \begin{pmatrix} 1+\mu & \mu^{1/2} \\ \mu^{1/2} & 1 \end{pmatrix} = A$$

whose trace is $2 + \mu$. So this product is pseudo-Anosov and its dilatation is the larger, λ , of the eigenvalues of A and the eigenvector is ${}^t(1, \sigma)$ with $\mu^{1/2} = (1 - \sigma^2)/\sigma$ and $\lambda = 1/\sigma^2$.

The heights and widths of the rectangles are coordinates of the μ -eigenvectors V of NN^t and V' of N^tN . We compute

$$V = \begin{pmatrix} 1 \\ \mu^2 - 5\mu + 1 \\ -2\mu^2 + 11\mu - 4 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

and

$$V' = \mu^{-1/2} N^t V = \mu^{-1/2} \begin{pmatrix} -\mu^2 + 6\mu - 2 \\ \mu^2 - 5\mu + 2 \\ \mu^2 - 5\mu + 1 \end{pmatrix} = \begin{pmatrix} v'_1 \\ v'_2 \\ v'_3 \end{pmatrix}$$

To show that F is prime it suffices to find 4 closed geodesics whose slopes have irrational cross ratio. We take $a = (1, 0)$, $b = (0, 1)$, $c = (-v'_2, v_1 + v_2)$, $d = (-v'_1 - v'_2, 2v_1 + v_2)$, where c and d connect second, respectively third, vertex on the lower left side in the figure with the upper right vertex. They are good closed geodesics on S based at the round vertex (to compute the angle at the round vertex, it helps first to identify the two edges labeled by z). The cross ratio is

$$\begin{aligned} \frac{[a, c][b, d]}{[a, d][b, c]} &= \frac{(v_1 + v_2)(v'_1 + v'_2)}{(2v_1 + v_2)v'_2} = \frac{(\mu^2 - 5\mu + 2)\mu}{(\mu^2 - 5\mu + 3)(\mu^2 - 5\mu + 2)} = \frac{\mu}{\mu^2 - 5\mu + 3} \\ &= \frac{1}{3\mu^2 - 17\mu + 10}, \end{aligned}$$

which is irrational (for the last equality use $\mu^3 - 6\mu^2 + 5\mu - 1 = 0$).

7 Invariable generation

In this section we discuss another application of Theorem 3.2. For this we need the version stated in Theorem 3.4.

7.1 Definitions and results

Following Dixon [1] a group G is *invariably generated* by a subset S of G if $G = \langle s^{g(s)} | s \in S \rangle$ for any choice of $g(s) \in G, s \in S$. The group G is *IG* if it is invariably generated by some subset S in G , or equivalently, if G is invariably generated by G . G is *FIG* if it is invariably generated by some finite subset of G . Kantor-Lubotzky-Shalev [2] prove that a linear group is FIG if and only if it is finitely generated and virtually solvable.

Gelander proves that every non-elementary hyperbolic group is not IG [8]. We generalize this result to acylindrically hyperbolic groups. A group G is *acylindrically hyperbolic* if it admits an acylindrical action on a hyperbolic space and G is not virtually cyclic [18]. Examples are non-elementary hyperbolic groups, $MCG(S_{g,p})$ except for the genus $g = 0$ and the number of punctures $p \leq 3$, and $Out(F_n)$ with $n \geq 2$ (cf. [18]).

We prove

Theorem 7.1. *If G is an acylindrically hyperbolic group, then G is not IG.*

In other words, G contains a proper subgroup such that any element in G is conjugate to some element in the subgroup.

7.2 Elementary facts from δ -hyperbolic spaces

Suppose G acts on a δ -hyperbolic space X .

For $g \in G$, define its *minimal translation length* by

$$\min(g) = \inf_{p \in X} |p - g(p)|$$

It is a well-known fact that if $\min(g) \geq 10\delta$ then g is hyperbolic. (To be precise, we assume $\delta > 0$ here.)

For $L > 0$ define a subset

$$X(g, L) = \{x \in X | |gx - x| \leq L\}$$

and put

$$M(g) = X(g, \min(g) + 1000\delta)$$

$M(g)$ is a g -invariant non-empty set. If g is hyperbolic, then $M(g)$ is contained in a Hausdorff neighborhood of an axis of g , $axis(g)$. This is an easy exercise and we leave it to the reader.

Lemma 7.2. *If g has a bounded orbit, then $\min(g) \leq 6\delta$.*

Proof. This is well known too. Let Z be the set of centers of the orbit of a point x by g . Z is invariant by g and its diameter is at most 6δ , therefore is contained in $X(g, 6\delta)$. \square

Lemma 7.3. *If $L \geq \min(g) + 1000\delta$, then $X(g, L)$ is 100δ -quasi-convex.*

Proof. We define a function on X as follows: $t_g(x) = |x - g(x)|$.

Case 1. $\min(g) \leq 10\delta$.

Fix $p \in X(g, 10\delta) \subset X(g, L)$. Suppose $x \in X(g, L)$ is given. We will show that $[p, x]$ is contained in the 20δ -neighborhood of $X(g, L)$. If $|x - p| \leq 100\delta$, then by triangle inequality, $t_g(z) \leq 210\delta$ for every $z \in [p, x]$, so that $[p, x] \subset X(g, L)$. So assume $|x - p| > 100\delta$. To compute $t_g(z)$ for $z \in [p, x]$, draw a triangle Δ for $p, x, g(x)$, and let $c \in [p, x]$ be a branch point of this triangle, i.e., the distance to each side of Δ from c is at most δ . Since $|p - g(p)| \leq 10\delta$, $t_g(z) \leq 20\delta$ for any point $z \in [p, c]$. For $z \in [c, x]$, $t_g(z)$ is roughly equal to $2d(c, z)$, with an additive error at most 20δ . Also it is roughly maximal at x on $[p, x]$. (Imagine the case that X is a tree and $p = g(p)$.) It follows that $[p, x]$ is contained in the 20δ -neighborhood of $X(g, L)$.

Now suppose another point $y \in X(g, L)$ is given. Then $[x, y]$ is contained in the δ -neighborhood of $[p, x] \cup [p, y]$, so that $[x, y]$ is contained in the 21δ -neighborhood of $X(g, L)$.

Case 2. $\min(g) \geq 10\delta$.

Then g is hyperbolic. To simplify the argument, let's assume that there is a geodesic axis for g . Then for any $x \in X$,

$$|t_g(x) - \{trans(g) + 2d(x, axis(g))\}| \leq 20\delta$$

To see this let $x' \in axis(g)$ be a nearest point from x . Then the Hausdorff distance between $[x, g(x)]$ and $[x, x'] \cup [x', g(x')] \cup [g(x'), g(x)]$ is at most 5δ , and the estimate follows.

It follows from the above estimate that if $x \in X(g, L)$, then $[x, x']$ is contained in the 20δ -neighborhood of $X(g, L)$. For $y \in X(g, L)$, let $y' \in$

$axis(g)$ be a nearest point from another point y to $axis(g)$. Then $[x, x'] \cup [x', y'] \cup [y', y]$ is contained in the 20δ -neighborhood of $X(g, L)$. But since $[x, y]$ is contained in the 5δ -neighborhood of $[x, x'] \cup [x', y'] \cup [y', y]$, $[x, y]$ is contained in the 25δ -neighborhood of $X(g, L)$.

The argument for the case that g has only a quasi-geodesic (with uniform quasi-geodesic constants depending only on δ) as an axis is similar and we only need to modify the constants in the argument. We omit the details. \square

Lemma 7.3 implies

Lemma 7.4. *$M(g)$ is 100δ -quasi-convex.*

Let ∂X denote the boundary at infinity of X . For a quasi-convex subset $Y \subset X$, let $\partial Y \subset \partial X$ be the boundary at infinity of Y .

Lemma 7.5. *If $p \in \partial M(g) \subset \partial X$, then $g(p) = p$.*

Proof. Let γ be a geodesic ray from a point in $M(g)$ that tends to p . Then the ray is contained in the 10δ -neighborhood of $M(g)$. So every point of the ray is moved by g by a bounded amount, therefore p is fixed by g . \square

When f is hyperbolic, the subgroup of elements in G that fix each point of $\partial(axis(f))$ is called the *elementary closure* of f , denoted by $E(f)$.

Lemma 7.6. *Assume the action of G is acylindrical on X . If g fixes one point in $\partial(axis(f))$, then $g \in E(f)$.*

Proof. Let γ be a half of $axis(f)$ that tends to the point fixed by g . Then $|x - gx|$ is bounded for $x \in \gamma$. Assume that γ tends to the direction that f translates $axis(f)$ (otherwise, we let $N < 0$ below). Then for any $N > 0$ and $x \in \gamma$, $|f^{-N}gf^N(x) - x|$ is bounded. Now by acylindricity (apply it to $x, y \in \gamma$ that are far from each other), there are only finitely many possibilities for $f^{-N}gf^N$, so g commutes with a nontrivial power of f . So g moves each point in $axis(f)$ by a bounded amount, therefore $g \in E(f)$. \square

Lemma 7.7. *Assume f is hyperbolic on X . If $g \notin E(f)$, then $\pi_{axis(f)}M(g)$ is bounded.*

Proof. Suppose not. Let p be a point in $\partial(axis(f))$ that the projection of $M(g)$ tends to.

We claim $p \in \partial M(g)$. This is because since both $axis(f)$ and $M(g)$ are quasi-convex, a half of $axis(f)$ to the direction of p is contained in a bounded neighborhood of $M(g)$.

So, by Lemma 7.5 $g(p) = p$, and by Lemma 7.6 g is in $E(f)$, a contradiction. \square

We will use the following result. It follows from the assumption that G contains a “hyperbolically embedded subgroup” that is non-degenerate, i.e., proper and infinite, see Theorem 1.2 in [18].

By a *Schottky subgroup* $F < G$ we mean a free subgroup such that an orbit map $F \rightarrow X$ is a quasi-isometric embedding.

Proposition 7.8 ([6, Theorem 6.14]). *Suppose the action of G is acylindrical and G is not virtually cyclic.*

Then G contains a unique maximal finite normal subgroup K and a Schottky subgroup F so that for every nontrivial $f \in F$ any element $g \in E(f)$ is either contained in K or has a nontrivial power that commutes with f . If K is trivial, $E(f)$ is cyclic.

When the order of g is $N < \infty$, we define

$$MM(g) = \cup_{0 < n < N} M(g^n).$$

This set is invariant by g , and contains $M(g)$. The following is obvious from the definition of the set $M(g^n)$ and $MM(g)$.

Lemma 7.9. *For any $x \in X - MM(g)$ and any non-trivial $h \in \langle g \rangle$, we have $|h(x) - x| > 1000\delta$.*

Remark 7.10. *Although we will not use this fact, we observe that if g has finite order N , then the set $MM(g)$ is 101δ -quasi-convex. This is because g has an orbit whose diameter is at most 6δ (see the proof of Lemma 7.2), and $MM(g) = \cup_{0 < n < N} M(g^n)$ is a union of 100δ -quasi-convex sets all of which contain the bounded orbit. Thus $MM(g)$ is 101δ -quasi-convex as desired. (Fix a point x from the bounded orbit. Then for any points $y, z \in MM(g)$, draw a δ -thin triangle for x, y, z . $[y, z]$ is in the δ -neighborhood of $[x, y] \cup [x, z]$.)*

Lemma 7.11. *Suppose there is a Schottky free subgroup $F < G$ such that any non-trivial $f \in F$ is hyperbolic and $E(f)$ is cyclic.*

Let $S = \{g_1, g_2, g_3, \dots\}$ be a (finite or infinite) set of non-trivial elements in G .

Then for any given $K > 0$ there is a set $S' = \{g'_i\}$ in G such that

- g_i and g'_i are conjugate for each i .
- For any $n > 0$, the sets $M(g'_1), \dots, M(g'_n)$ are K -separated, and moreover, this property holds if we replace $M(g'_k)$ with $MM(g'_k)$ when g'_k have finite order.

Proof. We first prepare a sequence of elements in F that we will use to conjugate g_i to g'_i . Take two elements $a, b \in F$ that produce a free subgroup of rank two. Put $f_i = ab^i, i \geq 1$.

We describe a geometric property we use about the sequence of elements. Fix a point $x \in X$. Given a constant $L > 0$, if we choose P sufficiently large, then for any $n > 0$, and any $P_i \geq P$, the following points are L -separated:

$$x, f_1^{P_1}(x), f_2^{P_2}(x), \dots, f_n^{P_n}(x).$$

This is an easy consequence of the property such that the embedding of the Cayley graph of F in X using the orbit of the point x is quasi-isometric to the image.

Note that the subsets in the above are L -separated if we replace the point x by a bounded set, possibly taking a larger constant for P .

To define g'_1, g'_2, g'_3, \dots , choose a sequence

$$1 \neq n_1 < n_2 < n_3 < \dots$$

such that for each i , $g_i \notin E(f_{n_i})$. This is clearly possible. Then by Lemma 7.7, the projection of $M(g_i)$ to $axis(f_{n_i})$ is bounded. Moreover, if the order of g_i is $N < \infty$, then $\langle g_i \rangle \cap E(f_{n_i}) = 1$, therefore the projection of $MM(g_i)$ to $axis(f_{n_i})$ is bounded since each $M(g_i^n), 0 < n < N$ has a bounded projection.

Now take a sequence of sufficiently large constants $L_i > 0$, depending on the given constant K , and put $g'_i = f_{n_i}^{L_i} g_i f_{n_i}^{-L_i}$. Then for each $n > 0$, the sets $M(g'_1), \dots, M(g'_n)$ are K -separated since $M(g'_i) = f_{n_i}^{L_i}(M(g_i))$. Also we can arrange so that the sets remain K -separated if we replace $M(g'_i)$ with $MM(g'_i)$ if the order of g'_i are finite, maybe for larger constants L_i . \square

Proof of Theorem 7.1.

Case 1. Assume G does not contain any non-trivial finite normal subgroup.

By Proposition 7.8, there is a Schottky subgroup $F < G$ such that any non-trivial element $f \in F$ is hyperbolic and $E(f)$ is cyclic.

Let $K > 0$ be a constant from Theorem 3.4 for $Q = 100\delta$. Let $\mathcal{C} = \{g_1, g_2, \dots\}$ be a set of all conjugacy classes of G except for the class for 1. Apply lemma 7.11 to the set \mathcal{C} and the constant K and obtain a new set $\mathcal{C}' = \{g'_i\}$. For each $i > 0$, $M(g'_i)$ is a non-empty, g'_i -invariant, 100δ -quasi-convex subset.

Now, for each $n > 1$, the assumptions (b) and (c) of Theorem 3.4 are satisfied by the subgroups $\langle g'_i \rangle, 1 \leq i \leq n$ and the sets $M(g'_i), 1 \leq i \leq n$. If g'_k has finite order, then take $M(g'_k) \subset MM(g'_k)$ as the desired neighborhood. Then by Lemma 7.11 they are K -separated, which implies (b) and (c).

We claim that (a) holds for g'_k that has finite order. But for any point $x \in X - MM(g'_k)$, and any non-trivial $h \in \langle g'_k \rangle$, we have $|h(x) - x| \geq 1000\delta$ (Lemma 7.9). This implies (a).

It now follows from Theorem 3.4 that the subgroup generated by g'_1, \dots, g'_n is the free product $\langle g'_1 \rangle * \dots * \langle g'_n \rangle$ for each $n > 0$.

To finish, first suppose that \mathcal{C}' is a finite set, $\{g'_1, \dots, g'_n\}$. Then $\langle g'_1 \rangle * \dots * \langle g'_n \rangle$ must be a proper subgroup of G (therefore G is not IG) since otherwise G contains infinitely many conjugacy classes, a contradiction (by our assumption, G is not virtually cyclic).

Second, we assume that \mathcal{C}' is an infinite set in the following. To argue by contradiction, assume that G is generated by \mathcal{C}' .

In $\langle g'_1 \rangle * \langle g'_2 \rangle * \langle g'_3 \rangle$, it is easy to choose elements g''_1, g''_2, g''_3 such that each g''_i is conjugate to g'_i , and the subgroup generated by g''_1, g''_2, g''_3 is a proper subgroup of $\langle g'_1 \rangle * \langle g'_2 \rangle * \langle g'_3 \rangle$.

Define \mathcal{C}'' from \mathcal{C}' by replacing g'_1, g'_2, g'_3 by g''_1, g''_2, g''_3 . \mathcal{C}'' contains all non-trivial conjugacy classes of G . We claim that the subgroup, G_1 , generated by \mathcal{C}'' is a proper subgroup in G (so G is not IG). To see that, define a quotient homomorphism from G to the group $H = \langle g'_1 \rangle * \langle g'_2 \rangle * \langle g'_3 \rangle$ by sending all $g'_i, i > 3$ to 1. Then the image of G_1 is a proper subgroup in H , so G_1 is a proper subgroup in G .

Case 2. Assume that G contains a non-trivial finite normal subgroup.

Let K be the maximal finite normal subgroup in G . Then $G' = G/K$ does not contain any non-trivial finite normal subgroup. Moreover G' is acylindrically hyperbolic group. Probably this fact is well known to specialists, and we postpone giving an argument till the end (Proposition 7.12).

By Case 1, G' contains a proper subgroup H' that contains all conjugacy

classes of G' . Let $H < G$ be the pull-back of H' by the quotient map $G \rightarrow G'$. Then H is a proper subgroup that contains all conjugacy classes of G , therefore G is not IG. \square

Proposition 7.12. *Let G be an acylindrically hyperbolic group and $N < G$ a finite normal subgroup. Then $G' = G/N$ is an acylindrically hyperbolic group.*

Proof. By assumption G acts on a hyperbolic space X such that the action is acylindrical and G is not virtually cyclic. Moreover we may assume that the action is co-compact. In fact, we may assume that X is a Cayley graph with a certain generating set, which is maybe infinite, [18, Theorem 1.2].

We will produce a new G -graph Y from X such that the kernel of the action contains N and that Y and X are quasi-isometric. This is a desired action for G' .

For each N -orbit of a vertex of X , we assign a vertex of Y . Note that G is transitive on the set of N -orbits of vertexes of X , so G acts transitively on the vertex set of Y . Now join two vertices of Y if the distance of the corresponding N -orbits in X is 1. Y is a connected G -graph and it is easy to check that Y and X are quasi-isometric (by the obvious map sending a vertex x of X to the vertex of Y corresponding to the orbit of x), so that Y is hyperbolic, and that the G -action on Y is acylindrical. By construction, N acts trivially on Y . \square

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